

# Boson Normal Ordering via Substitutions and Sheffer-type Polynomials

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**Abstract.** We solve the boson normal ordering problem for  $(q(a^\dagger)a + v(a^\dagger))^n$  with arbitrary functions  $q$  and  $v$  and integer  $n$ , where  $a$  and  $a^\dagger$  are boson annihilation and creation operators, satisfying  $[a, a^\dagger] = 1$ . This leads to exponential operators generalizing the shift operator and we show that their action can be expressed in terms of substitutions. Our solution is naturally related through the coherent state representation to the exponential generating functions of Sheffer-type polynomials. This in turn opens a vast arena of combinatorial methodology which is applied to boson normal ordering and illustrated by a few examples.

## 1. Introduction

Three decades ago Navon [1] and Katriel [2] observed that the normal ordering of quantum operators is a combinatorial problem. Navon considered general products of fermion creation  $a^\dagger$  and annihilation  $a$  operators and showed that such monomials can be expressed as sums of lower order normal products with coefficients which are combinatorial numbers called *rook numbers*. Katriel analysed the normally ordered form of  $(a^\dagger a)^n$  for the boson case  $[a, a^\dagger] = 1$  and arrived at a solution given in terms of the *Stirling numbers of the second kind*. Following these ideas we presented closed form solutions for the normally ordered form of Taylor expandable boson operator functions  $F(\hat{w}(a, a^\dagger))$  where  $\hat{w}(a, a^\dagger)$  is either a monomial in  $a$  and  $a^\dagger$  operators or a monomial in  $a + a^\dagger$  [3]. Using methods of combinatorial analysis we identified combinatorial numbers

appearing in such normally ordered expressions and described their properties. Recently Witschel [4] presented an alternative technique for this problem.

In this Letter we solve the problem of normally ordering expressions of the form  $(q(a^\dagger)a + v(a^\dagger))^n$  for arbitrary functions  $q(a^\dagger)$  and  $v(a^\dagger)$ . This generalizes results [5], [6] obtained in the mid 80's. Our considerations have been inspired by operational methods proposed by Dattoli *et al.* [7], [8]. Our approach connects the boson normal ordering problem to the notion of substitution groups and, if one employs the coherent state representation, to the theory of Sheffer-type polynomials appearing in various areas of combinatorics [9],[10].

## 2. Operational formulas via substitution group

We consider bosons satisfying  $[a, a^\dagger] = 1$ , and use the representation of the Heisenberg - Weyl algebra in terms of multiplication and derivative operators  $[\frac{d}{dx}, x] = 1$ . We start with the Taylor formula  $\exp\left(\lambda \frac{d}{dx}\right) F(x) = F(x + \lambda)$  and consider a generalization of the shift operator of the form  $E_{q,v}(\lambda) = \exp\left[\lambda \left(q(x) \frac{d}{dx} + v(x)\right)\right]$ , where  $q(x)$  and  $v(x)$  are arbitrary functions. We shall find a formula for  $E_{q,v}(\lambda)F(x)$ . In fact, as we will demonstrate, the following equality holds

$$\exp\left[\lambda \left(q(x) \frac{d}{dx} + v(x)\right)\right] F(x) = g(\lambda, x) \cdot F(T(\lambda, x)) \quad (1)$$

where

$$\frac{\partial T(\lambda, x)}{\partial \lambda} = q(T(\lambda, x)) , \quad T(0, x) = x , \quad (2)$$

$$\frac{\partial g(\lambda, x)}{\partial \lambda} = v(T(\lambda, x)) \cdot g(\lambda, x) , \quad g(0, x) = 1 . \quad (3)$$

We now exploit basic properties of Eqs.(1)-(3). First observe from Eq.(1) that the action of  $E_{q,v}(\lambda)$  on a function  $F(x)$  amounts to: a) a change of argument  $x \rightarrow T(\lambda, x)$  in  $F(x)$  which is in fact a substitution; b) multiplication by a prefactor  $g(\lambda, x)$  which we call a *prefunction*. We also see from Eq.(3) that  $g(\lambda, x) = 1$  for  $v(x) = 0$ . Finally, note that  $E_{q,v}(\lambda)$  with  $\lambda$  real generates an abelian, one-parameter group, implemented by Eq.(1); this gives the following group composition law for  $T(\lambda, x)$  and  $g(\lambda, x)$ :

$$\begin{aligned} T(\lambda + \theta, x) &= T(\theta, T(\lambda, x)), \\ g(\lambda + \theta, x) &= g(\lambda, x) \cdot g(\theta, T(\lambda, x)). \end{aligned} \quad (4)$$

In order to prove Eqs.(1)-(3) first recall the exponential mapping formula which for operators  $A$  and  $B$  states that

$$e^{\lambda A} B e^{-\lambda A} = B + \lambda[A, B] + \frac{\lambda^2}{2!}[A, [A, B]] + \dots . \quad (5)$$

Calculating appropriate commutators one can check that for arbitrary functions  $v(x)$  and  $G(x)$ , where  $x$  is the multiplication operator, the following operator relation holds†

$$e^{\lambda[\frac{d}{dx}+v(x)]}G(x)e^{-\lambda[\frac{d}{dx}+v(x)]} = G(x + \lambda). \quad (6)$$

Next, in the l.h.s. of Eq.(1) let us change the variables according to  $x = \tilde{f}(y)$  subject to a constraint

$$\frac{d}{dy}\tilde{f}(y) = q(\tilde{f}(y)), \quad (7)$$

which leads to

$$e^{\lambda[q(x)\frac{d}{dx}+v(x)]}F(x) = e^{\lambda[\frac{d}{dy}+v(\tilde{f}(y))]}F(\tilde{f}(y)). \quad (8)$$

We treat the r.h.s of above equation as an operator acting on a constant function 1 and we get

$$e^{\lambda[q(x)\frac{d}{dx}+v(x)]}F(x) = e^{\lambda[\frac{d}{dy}+v(\tilde{f}(y))]}F(\tilde{f}(y))e^{-\lambda[\frac{d}{dy}+v(\tilde{f}(y))]}e^{\lambda[\frac{d}{dy}+v(\tilde{f}(y))]} 1 \quad (9)$$

which using Eq.(6) gives

$$e^{\lambda[\frac{d}{dx}+v(x)]}F(x) = F(\tilde{f}(y + \lambda))\tilde{g}(\lambda, y) \quad (10)$$

where

$$\tilde{g}(\lambda, y) = e^{\lambda[\frac{d}{dx}+v(\tilde{f}(y))]} 1. \quad (11)$$

We can find the differential equation satisfied by  $\tilde{g}(\lambda, y)$

$$\begin{aligned} \frac{\partial \tilde{g}(\lambda, y)}{\partial \lambda} &= e^{\lambda[\frac{d}{dx}+v(\tilde{f}(y))]}v(\tilde{f}(y)) 1 \\ &= e^{\lambda[\frac{d}{dx}+v(\tilde{f}(y))]}v(\tilde{f}(y))e^{-\lambda[\frac{d}{dx}+v(\tilde{f}(y))]}e^{\lambda[\frac{d}{dx}+v(\tilde{f}(y))]} 1 \\ &= v(\tilde{f}(y) + \lambda)\tilde{g}(\lambda, y), \end{aligned} \quad (12)$$

with the use of Eq.(6) and definition Eq.(11). The initial condition is  $\tilde{g}(0, y) = 1$ .

If we identify

$$\begin{aligned} T(\lambda, x) &= \tilde{f}(\tilde{f}^{-1}(x) + \lambda) \\ g(\lambda, x) &= \tilde{g}(\lambda, \tilde{f}^{-1}(x)) \end{aligned} \quad (13)$$

then Eqs.(7), (10) and (12) give Eqs.(1)-(3) and so complete the proof. Note that formulas of the type Eq.(13) were used by G.A. Goldin in his investigations of current algebras [11] and more recently by G. Dattoli *et al* [8]. For a geometrical interpretation of Eq.(13) and the existence of the prefunction see also [14].

Here we list several applications of Eqs.(1)-(3) for some choices of  $q(x)$  and  $v(x)$ . Since Eqs.(2) and (3) are first order linear differential equations we shall simply write

† A word of explanation is appropriate when comparing Eqs.(1) and (6): Eq.(1) is an equality between functions, whereas Eq.(6) is an operator equation. Upon rewriting Eq.(6) as  $e^{\lambda[\frac{d}{dx}+v(x)]}G(x) = G(x + \lambda)e^{\lambda[\frac{d}{dx}+v(x)]}$  and by acting with it on a constant function 1 we obtain a particular case of Eq.(1) for  $q(x) = 1$  and  $g(\lambda, x) \equiv e^{\lambda[\frac{d}{dx}+v(x)]} 1$ . For the equation for  $g(\lambda, x)$  see below.

down their solutions without dwelling on details. First we treat the case of  $v(x) = 0$ , which implies  $g(\lambda, x) \equiv 1$ :

$$\text{Ex.1 :} \quad q(x) = x, \quad T(\lambda, x) = xe^\lambda \quad (14)$$

which gives  $\exp\left(\lambda x \frac{d}{dx}\right) F(x) = F(xe^\lambda)$ , a well known illustration of the Euler dilation operator  $\exp\left(\lambda x \frac{d}{dx}\right)$ .

$$\text{Ex.2 :} \quad q(x) = x^r, \quad r > 1, \quad T(\lambda, x) = \frac{x}{\left(1 - \lambda(r-1)x^{r-1}\right)^{\frac{1}{r-1}}} \quad (15)$$

The above examples were already considered in the literature [3], [7], [12], [14].

We shall go on to examples of  $v(x) \neq 0$  leading to nontrivial prefunctions:

$$\begin{aligned} \text{Ex.3 :} \quad & q(x) = 1; \quad v(x) = \text{arbitrary}, \\ & T(\lambda, x) = x + \lambda, \\ & g(\lambda, x) = \exp\left[\int_0^\lambda du \, v(x+u)\right], \end{aligned} \quad (16)$$

$$\begin{aligned} \text{Ex.4 :} \quad & q(x) = x; \quad v(x) = x^2; \\ & T(\lambda, x) = xe^\lambda, \\ & g(\lambda, x) = \exp\left[\frac{x^2}{2}(e^{2\lambda} - 1)\right], \end{aligned} \quad (17)$$

$$\begin{aligned} \text{Ex.5 :} \quad & q(x) = x^r, \quad r > 1, \quad v(x) = x^s; \\ & T(\lambda, x) = \frac{x}{\left(1 - \lambda(r-1)x^{r-1}\right)^{\frac{1}{r-1}}}, \\ & g(\lambda, x) = \exp\left[\frac{x^{s-r+1}}{1-r} \left(\frac{1}{\left(1 - \lambda(r-1)x^{r-1}\right)^{\frac{s-r+1}{r-1}}} - 1\right)\right], \end{aligned} \quad (18)$$

Closer look at above examples (or at any other example which the reader could easily construct) indicates that the group property Eq.(4) may be true only if  $\lambda$  satisfies certain restrictions arising from the requirement that  $T(\lambda, x)$  and  $g(\lambda, x)$  remain real-valued functions. Evidently any result obtained with Eq.(1)-(3) should be examined in this respect and resulting restrictions on  $\lambda$  be kept in mind. In general we can say that the group property Eq.(4) may be valid only locally [13],[14].

### 3. Implications for boson normal ordering

The results elaborated above will be used now to treat the problem of the normal ordering of operator functions of boson operators. For a general function  $F(a, a^\dagger)$  its normally ordered form  $\mathcal{N}[F(a, a^\dagger)] \equiv F(a, a^\dagger)$  is obtained by moving all the annihilation operators  $a$  to the right, using the commutation relations. We may additionally define the operation  $: G(a, a^\dagger) :$  which means normally order  $G(a, a^\dagger)$  *without* taking into account the commutation relations. Using the latter operation the normal ordering problem is solved for  $F(a, a^\dagger)$  if we are able to find an operator  $G(a, a^\dagger)$  for which  $F(a, a^\dagger) =: G(a, a^\dagger) :$  is satisfied. To obtain the normally ordered form of functions of

boson operators is a nontrivial task even for simple functions. But it is nevertheless worthwhile, because the knowledge of  $\mathcal{N}[F(a, a^\dagger)]$  is very useful in practice. It allows one to evaluate easily the expectation values of  $F(a, a^\dagger)$  in such important classes of states as the vacuum and coherent states, the latter defined for a complex number  $z$  as  $|z\rangle = \exp(-\frac{|z|^2}{2}) \sum_{n=0}^{\infty} \frac{z^n}{\sqrt{n!}} |n\rangle$ , (with  $a^\dagger a |n\rangle = n |n\rangle$ ,  $\langle n | n'\rangle = \delta_{n,n'}$ ), and satisfying  $a|z\rangle = z|z\rangle$ .

Here we will apply the results of Section 2 and obtain  $\mathcal{N}[e^{\lambda[q(a^\dagger)a+v(a^\dagger)]}]$  for arbitrary functions  $q(x)$  and  $v(x)$ . First note that

$$[q(a^\dagger)a + v(a^\dagger)]^n = h_n(a^\dagger) + \sum_{k=1}^n f_{nk}(a^\dagger)a^k \quad (19)$$

with appropriately defined  $f_{n,k}$  and  $h_n$ . Consequently one obtains

$$e^{\lambda[q(a^\dagger)a+v(a^\dagger)]} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left( h_n(a^\dagger) + \sum_{k=1}^n f_{nk}(a^\dagger)a^k \right), \quad (20)$$

whose matrix elements between the coherent states  $|z\rangle$  and  $|z'\rangle$  are

$$\langle z' | e^{\lambda[q(a^\dagger)a+v(a^\dagger)]} | z \rangle = \langle z' | z \rangle \left( 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left( h_n(z'^*) + \sum_{k=1}^n f_{nk}(z'^*)z^k \right) \right). \quad (21)$$

The next objective is to evaluate the sum on the r.h.s. of Eq.(21). We do so using the representation  $a^\dagger \rightarrow x$  and  $a \rightarrow \frac{d}{dx}$  and rewrite Eq.(20) as

$$e^{\lambda[q(x)\frac{d}{dx}+v(x)]} = 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left( h_n(x) + \sum_{k=1}^n f_{nk}(x) \frac{d^k}{dx^k} \right). \quad (22)$$

Next, by acting with this operator identity on  $e^{yx}$  we have

$$e^{\lambda[q(x)\frac{d}{dx}+v(x)]} e^{yx} = \left[ 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left( h_n(x) + \sum_{k=1}^n f_{nk}(x) y^k \right) \right] e^{yx}. \quad (23)$$

Eq.(1) allows one to rewrite the l.h.s. of the above equation (with  $T(\lambda, x)$  and  $g(\lambda, x)$  solutions of Eqs.(2) and (3)) as

$$e^{\lambda[q(x)\frac{d}{dx}+v(x)]} e^{yx} = g(\lambda, x) e^{yT(\lambda, x)}, \quad (24)$$

which, if put into Eq.(23), implies

$$g(\lambda, x) e^{y[T(\lambda, x)-x]} = \left[ 1 + \sum_{n=1}^{\infty} \frac{\lambda^n}{n!} \left( h_n(x) + \sum_{k=1}^n f_{nk}(x) y^k \right) \right]. \quad (25)$$

Using the last equality on the r.h.s of (21) we obtain

$$\langle z' | e^{\lambda[q(a^\dagger)a+v(a^\dagger)]} | z \rangle = \langle z' | z \rangle g(\lambda, z'^*) e^{[T(\lambda, z'^*)-z'^*]z}. \quad (26)$$

We now apply the crucial property of the coherent state representation (see [15] and [16]). This is that if for an arbitrary operator  $F(a, a^\dagger)$  we have

$$\langle z' | F(a, a^\dagger) | z \rangle = \langle z' | z \rangle G(z'^*, z) \quad (27)$$

then the normally ordered form of  $F(a, a^\dagger)$  is given by

$$\mathcal{N}[F(a, a^\dagger)] =: G(a^\dagger, a) :. \quad (28)$$

Eqs.(26) and (28) then provide the central result

$$\mathcal{N}[e^{\lambda[q(a^\dagger)a+v(a^\dagger)]}] =: g(\lambda, a^\dagger) e^{[T(\lambda, a^\dagger)-a^\dagger]a} :, \quad (29)$$

being an operator identity in which the functions  $g$ ,  $T$  are found through Eqs.(2) and (3).

#### 4. Normal ordering and Sheffer-type polynomials

We now investigate in more detail the properties of Eq.(26). In particular we shall identify its inherent characteristic polynomial structures. Using the representation of coherent states in terms of the Glauber displacement operator  $D(z)$

$$|z\rangle = D(z)|0\rangle = \exp(z a^\dagger - z^* a)|0\rangle = e^{-|z|^2/2} e^{z a^\dagger} |0\rangle, \quad (30)$$

where  $|0\rangle$  is the Fock vacuum, we rewrite the l.h.s of Eq.(26) as

$$\begin{aligned} \langle z' | e^{\lambda[q(a^\dagger)a + v(a^\dagger)]} | z \rangle &= \\ &= e^{-1/2(|z'|^2 + |z|^2)} \langle 0 | e^{z'^* a} e^{z a^\dagger} e^{-z a^\dagger} e^{\lambda[q(a^\dagger)a + v(a^\dagger)]} e^{z a^\dagger} | 0 \rangle. \end{aligned} \quad (31)$$

Next we apply the exponential mapping formula Eq.(5) in order to get

$$\begin{aligned} e^{-z a^\dagger} e^{\lambda[q(a^\dagger)a + v(a^\dagger)]} e^{z a^\dagger} &= e^{\lambda[q(a^\dagger)(a+z) + v(a^\dagger)]}, \\ e^{z'^* a} e^{z a^\dagger} &= e^{z a^\dagger} e^{z'^* a} e^{z'^* z} \end{aligned} \quad (32)$$

which, if put into Eq. (31), give (with  $\langle z' | z \rangle = \exp(z'^* z - \frac{1}{2}|z'|^2 - \frac{1}{2}|z|^2)$ )

$$\langle z' | e^{\lambda[q(a^\dagger)a + v(a^\dagger)]} | z \rangle = \langle z' | z \rangle e^{\frac{1}{2}|z'|^2} \langle z' | e^{\lambda[q(a^\dagger)(a+z) + v(a^\dagger)]} | 0 \rangle. \quad (33)$$

If we expand the exponential on the r.h.s. of Eq.(33) as a Taylor series in  $\lambda$ , then the expansion coefficients for any fixed  $z'$  are polynomials in  $z$ .

Properties of such polynomials may be clarified when one recalls the definition of the family of Sheffer-type polynomials  $S_n(z)$  [9]. The latter are defined through the *exponential generating function* (egf) as

$$1 + \sum_{n=1}^{\infty} S_n(z) \frac{\lambda^n}{n!} = A(\lambda) e^{z B(\lambda)} \quad (34)$$

where functions  $A(\lambda)$  and  $B(\lambda)$  satisfy:  $A(0) = 1$  and  $B(0) = 0$ ,  $B'(0) \neq 0$ . Many properties of these polynomials may be elucidated by means of the so called *umbral calculus* [9], [17], [18] which provides us with numerous interesting applications.

Returning to normal order, recall that the coherent state expectation value of Eq.(29) is given by Eq.(26). When one fixes  $z'$  and takes  $\lambda$  and  $z$  as indeterminates, then the r.h.s. of Eq.(26) may be read off as an egf of Sheffer-type polynomials defined by Eq.(34). The correspondence is given by

$$A(\lambda) = g(\lambda, z'^*), \quad (35)$$

$$B(\lambda) = [T(\lambda, z'^*) - z'^*]. \quad (36)$$

This allows us to make the statement that the coherent state expectation value  $\langle z' | \dots | z \rangle$  of the operator  $\exp[\lambda(q(a^\dagger)a + v(a^\dagger))]$  for any fixed  $z'$  yields (up to the overlapping factor  $\langle z' | z \rangle$ ) the egf of a certain sequence of Sheffer-type polynomials in the variable  $z$  given by Eqs.(35) and (36). The above construction establishes the connection between the coherent state representation of the operator  $\exp[\lambda(q(a^\dagger)a + v(a^\dagger))]$  and a family of Sheffer-type polynomials  $S_n^{q,v}(z)$  related to  $q$  and  $v$  through

$$\langle z' | e^{\lambda[q(a^\dagger)a + v(a^\dagger)]} | z \rangle = \langle z' | z \rangle \left( 1 + \sum_{n=1}^{\infty} S_n^{(q,v)}(z) \frac{\lambda^n}{n!} \right), \quad (37)$$

where explicitly (again for  $z'$  fixed):

$$\begin{aligned} S_n^{(q,v)}(z) &= \langle z'|z \rangle^{-1} \langle z'| \left[ q(a^\dagger)a + v(a^\dagger) \right]^n |z \rangle \\ &= e^{\frac{1}{2}|z'|^2} \langle z'| \left[ q(a^\dagger)(a+z) + v(a^\dagger) \right]^n |0 \rangle. \end{aligned} \quad (38)$$

We observe that Eq.(38) is an extension of the seminal formula of Katriel [2],[6] where  $v(x) = 0$  and  $q(x) = x$ . The Sheffer-type polynomials are in this case exponential polynomials [9] expressible through the Stirling numbers of the second kind.

Having established relations leading from the normal ordering problem to Sheffer-type polynomials we may consider the reverse approach. Indeed, it turns out that for any Sheffer-type sequence generated by  $A(\lambda)$  and  $B(\lambda)$  one can find functions  $q(x)$  and  $v(x)$  such that the coherent state expectation value  $\langle z'| \exp \left[ \lambda(q(a^\dagger)a + v(a^\dagger)) \right] |z \rangle$  results in a corresponding egf of Eq.(34) in indeterminates  $z$  and  $\lambda$  (up to the overlapping factor  $\langle z'|z \rangle$  and  $z'$  fixed). Appropriate formulas can be derived from Eqs.(35) and (36) by substitution into Eqs.(2) and (3):

$$q(x) = B'(B^{-1}(x - z'^*)), \quad (39)$$

$$v(x) = \frac{A'(B^{-1}(x - z'^*))}{A(B^{-1}(x - z'^*))}. \quad (40)$$

One can check that this choice of  $q(x)$  and  $v(x)$ , if inserted into Eqs. (2) and (3), results in

$$T(\lambda, x) = B(\lambda + B^{-1}(x - z'^*)) + z'^*, \quad (41)$$

$$g(\lambda, x) = \frac{A(\lambda + B^{-1}(x - z'^*))}{A(B^{-1}(x - z'^*))}, \quad (42)$$

which reproduce

$$\langle z'| e^{\lambda[q(a^\dagger)a + v(a^\dagger)]} |z \rangle = \langle z'|z \rangle A(\lambda) e^{zB(\lambda)}. \quad (43)$$

The result summarized in Eqs.(35) and (36) and in their 'dual' forms Eqs.(39)-(42), provide us with a considerable flexibility in conceiving and analyzing a large number of examples.

## 5. Combinatorial structures

In this section we will work out examples illustrating how the egf ( $= \sum_{n=0}^{\infty} a(n) \frac{x^n}{n!}$ ) of certain combinatorial sequences  $a(n)$ ,  $n = 0, 1, 2, \dots$  appear naturally in the context of boson normal ordering. To this end we shall assume specific forms of  $q(x)$  and  $v(x)$  thus specifying the operator that we exponentiate. We then give solutions to Eqs.(2) and (3) and subsequently through Eqs.(35) and (36) we shall write the egf of combinatorial sequences whose interpretation will be given.

a) Choose  $q(x) = x^r$ ,  $r > 1$  (integer),  $v(x) = 0$  (which implies  $g(\lambda, x) = 1$ ). Then (see Eq.(15))  $T(\lambda, x) = x[1 - \lambda(r-1)x^{r-1}]^{\frac{1}{1-r}}$ . This gives

$$\mathcal{N} \left[ e^{\lambda(a^\dagger)^r a} \right] = : \exp \left[ \left( \frac{a^\dagger}{(1 - \lambda(r-1)(a^\dagger)^{r-1})^{\frac{1}{r-1}}} - 1 \right) a \right] : \quad (44)$$

as the normally ordered form. We now take  $z' = 1$  in Eqs.(2) and (3) and from Eq.(26) one has

$$\langle 1|z\rangle^{-1}\langle 1|e^{\lambda(a^\dagger)^ra}|z\rangle = \exp\left[z\left(\frac{1}{(1-\lambda(r-1))^{\frac{1}{r-1}}}-1\right)\right], \quad (45)$$

which for  $z = 1$  generates the following sequences:

$$r = 2 : \quad a(n) = 1, 1, 3, 13, 73, 501, 451, \dots \quad (46)$$

$$r = 3 : \quad a(n) = 1, 1, 4, 25, 211, 2236, 28471, \dots, \text{ etc.} \quad (47)$$

These sequences are enumerating  $r$ -ary forests [3], [19], [20], [21].

b) For  $q(x) = x \ln(ex)$  and  $v(x) = 0$  (implying  $g(\lambda, x) = 1$ ) which give  $T(\lambda, x) = e^{e^\lambda - 1}x^{e^\lambda}$ , this corresponds to

$$\mathcal{N}\left[e^{\lambda a^\dagger \ln(ea^\dagger)a}\right] = : \exp\left[\left(e^{e^\lambda - 1}(a^\dagger)^{e^\lambda} - 1\right)a\right] : , \quad (48)$$

whose coherent state matrix element with  $z' = 1$  is equal to

$$\langle 1|z\rangle^{-1}\langle 1|e^{\lambda a^\dagger \ln(ea^\dagger)a}|z\rangle = \exp\left[z\left(e^{e^\lambda - 1} - 1\right)\right], \quad (49)$$

which for  $z = 1$  generates  $a(n) = 1, 1, 3, 12, 60, 385, 2471, \dots$  corresponding to partitions of partitions [19], [20], [21].

The following two examples will refer to the reverse procedure, see Eqs.(39)-(42). We choose first a Sheffer-type egf and deduce  $q(x)$  and  $v(x)$  associated with it.

c)  $A(\lambda) = \frac{1}{1-\lambda}$ ,  $B(\lambda) = \lambda$ , see Eq.(34). This egf for  $z = 1$  counts the number of arrangements  $a(n) = n! \sum_{k=0}^n \frac{1}{k!} = 1, 2, 5, 65, 326, 1957, \dots$  of the set of  $n$  elements [22]. The solutions of Eqs.(39) and (40) are:  $q(x) = 1$  and  $v(x) = \frac{1}{2-x}$ . In terms of bosons it corresponds to

$$\mathcal{N}\left[e^{\lambda\left(a+\frac{1}{2-a^\dagger}\right)}\right] = : \frac{2-a^\dagger}{2-a^\dagger-\lambda}e^{\lambda a} : = \frac{2-a^\dagger}{2-a^\dagger-\lambda}e^{\lambda a}. \quad (50)$$

d) For  $A(\lambda) = 1$  and  $B(\lambda) = 1 - \sqrt{1-2\lambda}$  one gets the egf of the Bessel polynomials [23]. For  $z = 1$  they enumerate special paths on a lattice [24]. The corresponding sequence is  $a(n) = 1, 1, 7, 37, 266, 2431, \dots$ . The solutions of Eqs.(39) and (40) are:  $q(x) = \frac{1}{2-x}$  and  $v(x) = 0$ . It corresponds to

$$\mathcal{N}\left[e^{\lambda\frac{1}{2-a^\dagger}a}\right] = : e^{\left(1-\sqrt{(2-a^\dagger)-2\lambda}\right)a} : \quad (51)$$

in the boson formalism.

These examples show that any combinatorial structure which can be described by a Sheffer-type egf can be cast in boson language. This gives rise to a large number of formulas of the type Eqs.(50) and (51) which are important for physical applications. Many other examples can be worked out in detail and will be listed in a subsequent publication.



## 6. Conclusions

The formula of Eq.(29) is the key result of the present investigation. It allows one to obtain normal ordering for a vast range of problems involving one annihilation operator (or one creation operator through hermitian conjugation). We have exploited the connection between the Eq.(29) and the egf of Sheffer-type polynomials and have given examples of combinatorial structures described by particular choices of  $q(x)$  and  $v(x)$ .

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